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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\frac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{(4n^2-1)^2} = \frac{1}{8} \left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right).$$

When $n=1, 3, 5, 7, \dots$

$$\begin{aligned} \sum \frac{n}{(4n^2-1)^2} &= \frac{1}{8} \left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \dots - \frac{1}{3^2} - \frac{1}{7^2} - \frac{1}{11^2} - \dots \right) \\ &= \frac{1}{8} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) - \frac{1}{4} \left(\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots \right) \\ &= \frac{\pi^2}{64} - \frac{1}{4} (.1579 + \dots) = \frac{\pi^2}{64} - \frac{1}{4} \cdot \frac{2\pi^2}{125} \text{ nearly, } = \frac{\pi^2}{64} - \frac{\pi^2}{250} = \frac{93\pi^2}{8000}. \end{aligned}$$

We may also write

$$\sum \frac{n}{(4n^2-1)^2} = \sum_{m=1}^{\infty} \frac{2m-1}{(16m^2-16m+3)^2} = \frac{1}{8} \int_0^1 \frac{\tan^{-1} x}{x} dx.$$

This series is discussed by William E. Heal in Vol. IX, pp. 47-49, of the *MONTHLY*.*

Similar approximations were obtained by S. A. Corey, G. W. Greenwood, and J. Scheffer.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

*See also an article entitled "Note on the Numerical Transcendentals S_n and $s_n = S_n - 1$," by Professor W. Woolsey Johnson, in the current *Bulletin of the American Mathematical Society*.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

According to a method due to Euler the series

$$S = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \dots$$

may be converted into a continued fraction thus: Putting

$$S_1 = \frac{1}{B} - \frac{1}{C} + \frac{1}{D} - \frac{1}{E} + \dots \quad S_2 = \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \frac{1}{F} + \dots$$

$$S_3 = \frac{1}{D} - \frac{1}{E} + \frac{1}{F} - \frac{1}{G} + \dots \text{ etc., we get}$$

$$S = \frac{1}{A} - S_1 = \frac{1 - A S_1}{A}; \quad \therefore \frac{1}{S} = \frac{A}{1 - A S_1} = A + \frac{A^2 S_1}{1 - A S_1} = A + \frac{A^2}{-A + (1/S_1)}, \text{etc.}$$

$$\text{Thus, } S = \frac{1}{A + \frac{A^2}{B - A + \frac{B^2}{C - B + \frac{C^2}{C - D + \frac{D^2}{E - D + \dots}}}}} \dots \quad \dots \text{(I).}$$

Since $\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ we get, substituting in (I), $A=1$, $B=3$, $C=5$, $D=1$, etc.,

$$\frac{1}{4}\pi = \frac{1}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \dots}}}}$$

To convert the series $\frac{1}{a} - \frac{1}{ab} + \frac{1}{abc} - \frac{1}{abcd} + \dots$ into a continued fraction, we put in (I), $A=a$, $B=ab$, $C=abc$, $D=abcd$, etc., and thus we obtain

$$\frac{1}{a} - \frac{1}{ab} + \frac{1}{abc} - \frac{1}{abcd} + \dots = \frac{1}{a + \frac{a}{b - 1 + \frac{b}{c - 1 + \frac{c}{d - 1 + \dots}}}} \quad \dots \text{(II).}$$

To convert $\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \frac{1}{abcd} + \dots$ into a continued fraction, we have in (II) only to put $-b$, $-c$, $-d$, \dots for b , c , d , \dots and thus we get

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \frac{1}{abcd} + \dots = \frac{1}{a - \frac{a}{b + 1 - \frac{b}{c + 1 - \frac{c}{d + 1 - \dots}}}} \quad \dots \text{(III).}$$

Putting $a=2, b=3, c=4, d=5, \dots$ we get

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots = \frac{1}{2} - \frac{2}{4} - \frac{3}{5} - \frac{4}{6} - \frac{5}{7} - \dots$$

$$\text{Hence } e = 2 + \frac{1}{2} - \frac{2}{4} - \frac{3}{5} - \frac{4}{6} - \frac{5}{7} - \dots$$

Also solved by G. W. Greenwood, and G. B. M. Zerr.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4 - 4a_1a_3 + 3a_2^2)$ of the binary quartic $a_0x^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$ in terms of roots of the latter.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It can be shown that, if

$$\begin{aligned} a_0x^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 - 4a_3x_1x_2^3 + a_4x_2^4 \\ \equiv a_0(x_1^2 + 2px_1x_2 + qx_2^2)(x_1^2 + 2p'x_1x_2 + q'x_2^2), \end{aligned}$$

then $4\theta^3 - I\theta + J = 0$, where $I = a_0a_4 - 4a_1a_3 + 3a_2^2$, and $\theta = a_2 - a_0pp'$.

Let β, γ be the roots of $x_1^2 + 2px_1x_2 + qx_2^2 = 0$, and α, δ the roots of $x_1^2 + 2p'x_1x_2 + q'x_2^2 = 0$. Then

$$\theta = \frac{a_0}{6} \Sigma \beta\gamma - \frac{a_0}{4}(\beta + \gamma)(\alpha + \delta) = \frac{a_0}{12}(v - w),$$

where $u = (\beta - \gamma)(\alpha - \delta)$, $v = (\gamma - \alpha)(\beta - \delta)$, $w = (\alpha - \beta)(\gamma - \delta)$. The roots of the reduced cubic are therefore,

$$\frac{a_0}{12}(u - v), \frac{a_0}{12}(v - w), \frac{a_0}{12}(w - u).$$

It is easily found that $u + v + w = 0$. Consequently, $\Sigma vw = -\frac{1}{2} \Sigma u^2$, and

$$\frac{I}{4} = - \Sigma \theta_1 \theta_2 = - \frac{a_0^2}{144} \Sigma (uv - uw + vw - v^2) = \frac{a_0^2}{144} \cdot \frac{3}{2} \Sigma u^2.$$

Hence $I = \frac{a_0^2}{24} \Sigma u^2$, where u, v, w have the values given above.

Also solved by G. B. M. Zerr.